

Exercices corrigés.

Calculer la somme $S = 1 + \frac{\cos x}{\cos x} + \frac{\cos 2x}{\cos^2 x} + \frac{\cos 3x}{\cos^3 x} + \dots + \frac{\cos nx}{\cos^n x}$

Résolution

Posons

$$A = 1 + \frac{\cos x}{\cos x} + \frac{\cos 2x}{\cos^2 x} + \frac{\cos 3x}{\cos^3 x} + \dots + \frac{\cos nx}{\cos^n x} \quad \text{et} \quad B = \frac{\sin x}{\cos x} + \frac{\sin 2x}{\cos^2 x} + \frac{\sin 3x}{\cos^3 x} + \dots + \frac{\sin nx}{\cos^n x}$$

Calculons $S = A + iB$

$$\begin{aligned} A + iB &= 1 + \frac{\cos x}{\cos x} + \frac{\cos 2x}{\cos^2 x} + \frac{\cos 3x}{\cos^3 x} + \dots + \frac{\cos nx}{\cos^n x} + i \left(\frac{\sin x}{\cos x} + \frac{\sin 2x}{\cos^2 x} + \frac{\sin 3x}{\cos^3 x} + \dots + \frac{\sin nx}{\cos^n x} \right) \\ &= 1 + \frac{\cos x}{\cos x} + \frac{\cos 2x}{\cos^2 x} + \frac{\cos 3x}{\cos^3 x} + \dots + \frac{\cos nx}{\cos^n x} + \frac{i \sin x}{\cos x} + \frac{i \sin 2x}{\cos^2 x} + \frac{i \sin 3x}{\cos^3 x} + \dots + \frac{i \sin nx}{\cos^n x} \\ &= 1 + \frac{\cos x}{\cos x} + \frac{i \sin x}{\cos x} + \frac{\cos 2x}{\cos^2 x} + \frac{i \sin 2x}{\cos^2 x} + \frac{\cos 3x}{\cos^3 x} + \frac{i \sin 3x}{\cos^3 x} + \dots + \frac{\cos nx}{\cos^n x} + \frac{i \sin nx}{\cos^n x} \\ &= 1 + \frac{\cos x + i \sin x}{\cos x} + \frac{\cos 2x + i \sin 2x}{\cos^2 x} + \frac{\cos 3x + i \sin 3x}{\cos^3 x} + \dots + \frac{\cos nx + i \sin nx}{\cos^n x} \end{aligned}$$

Posons $\cos x + i \sin x = e^{ix}$

$$\begin{aligned} &= 1 + \frac{e^{ix}}{\cos x} + \frac{e^{i2x}}{\cos^2 x} + \frac{e^{i3x}}{\cos^3 x} + \dots + \frac{e^{inx}}{\cos^n x} \\ &= 1 + \frac{e^{ix}}{\cos x} + \left(\frac{e^{ix}}{\cos x} \right)^2 + \left(\frac{e^{ix}}{\cos x} \right)^3 + \dots + \left(\frac{e^{ix}}{\cos x} \right)^n \end{aligned}$$

On obtient donc les $n + 1$ premiers termes d'une suite géométrique de premier terme 1 et de raison $\frac{e^{ix}}{\cos x}$

$$\begin{aligned} A + iB &= \frac{1 - \left(\frac{e^{ix}}{\cos x} \right)^{n+1}}{1 - \frac{e^{ix}}{\cos x}} \\ &= \frac{1 - \frac{e^{i(n+1)x}}{\cos^{n+1} x}}{1 - \frac{e^{ix}}{\cos x}} = \frac{\cos^{n+1} x - \cos(n+1)x - i \sin(n+1)x}{\cos^{n+1} x} \\ &= \frac{1 - \frac{e^{ix}}{\cos x}}{\cos x} = \frac{\cos x - \cos x - i \sin x}{\cos x} \end{aligned}$$

$$\begin{aligned}
&= \frac{\cos^{n+1} x - \cos(n+1)x - i \sin(n+1)x}{\cos^{n+1} x} \times \frac{\cos x}{-i \sin x} \\
&= \frac{i \cos^{n+2} x - i \cos x \cos(n+1)x + \cos x \sin(n+1)x}{\sin x \cos^{n+1} x} \\
&= \frac{\cos x \sin(n+1)x}{\sin x \cos^{n+1} x} + i \frac{\cos^{n+2} x - \cos x \cos(n+1)x}{\sin x \cos^{n+1} x} \\
A + iB &= \frac{\cos x \sin(n+1)x}{\sin x \cos^{n+1} x} + i \frac{\cos^{n+2} x - \cos x \cos(n+1)x}{\sin x \cos^{n+1} x}
\end{aligned}$$

D'où par identification, on pose

$$\begin{cases}
A = \frac{\cos x \sin(n+1)x}{\sin x \cos^{n+1} x} \\
B = \frac{\cos^{n+2} x - \cos x \cos(n+1)x}{\sin x \cos^{n+1} x}
\end{cases}$$

$$A = 1 + \frac{\cos x}{\cos x} + \frac{\cos 2x}{\cos^2 x} + \frac{\cos 3x}{\cos^3 x} + \dots + \frac{\cos nx}{\cos^n x} = \frac{\cos x \sin(n+1)x}{\sin x \cos^{n+1} x}$$

$$1 + \frac{\cos x}{\cos x} + \frac{\cos 2x}{\cos^2 x} + \frac{\cos 3x}{\cos^3 x} + \dots + \frac{\cos nx}{\cos^n x} = \frac{\sin(n+1)x}{\sin x \cos^n x}$$

Par :

Nkeuna Ngueliako georges

PLEG – Informaticien

Lycée Bilingue de Nylon Brazzaville Douala - Cameroun