

**Exercices corrigés.**

Montrer que  $\int_0^{\frac{\pi}{4}} \left( \int_0^x \sin t \cos t dt \right) dx = \frac{15\pi - 44}{1152}$

**Résolution**

Posons  $I = \int_0^{\frac{\pi}{4}} \left( \int_0^x \sin t \cos t dt \right) dx = \frac{15\pi - 44}{1152}$

Considérons l'intégrale  $J = \int_0^x \sin t \cos t dt$

$$J = \int_0^x \sin^5 t \cos t dt = \left[ \frac{\sin^6 t}{6} \right]_0^x = \frac{\sin^6 x}{6}, \text{ car } \sin^5 t \cos t dt \text{ est de la forme } u'u^5 \text{ dont une}$$

primitive est  $\frac{u^6}{6} + k$

Donc  $I = \int_0^{\frac{\pi}{4}} \frac{\sin^6 x}{6} dx = \frac{1}{6} \int_0^{\frac{\pi}{4}} \sin^6 x dx$

Linéarisons  $\sin^6 x$

Posons  $z = \cos \theta + i \sin \theta$

$$\sin \theta = \frac{z - \bar{z}}{2i} \Rightarrow \sin^6 \theta = \left( \frac{z - \bar{z}}{2i} \right)^6 = \left( \frac{1}{2i} \right)^6 (z - \bar{z})^6$$

Développons  $(z - \bar{z})^6$

$$\begin{aligned} (z - \bar{z})^6 &= (z + (-\bar{z}))^6 = z^6 + 6z^5(-\bar{z}) + 15z^4(-\bar{z})^2 + 20z^3(-\bar{z})^3 + 15z^2(-\bar{z})^4 + 6z(-\bar{z})^5 + (-\bar{z})^6 \\ &= z^6 + \bar{z}^6 + 6z\bar{z}(z^4 + \bar{z}^4) + 15(z\bar{z})^2(z^2 + \bar{z}^2) + 20z^3(-\bar{z})^3 \\ &= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20 \end{aligned}$$

D'om on tire que  $\sin^6 x = -\frac{1}{64}(2\cos 6x - 12\cos 4x + 30\cos 2x - 20)$

$$= -\frac{1}{32}\cos 6x + \frac{12}{64}\cos 4x - \frac{30}{64}\cos 2x + \frac{20}{64}$$

$$= -\frac{1}{32}\cos 6x + \frac{6}{32}\cos 4x - \frac{15}{32}\cos 2x + \frac{10}{32}$$

$$I = \frac{1}{6} \int_0^{\frac{\pi}{4}} \sin^6 x dx = \frac{1}{6} \int_0^{\frac{\pi}{4}} \left( -\frac{1}{32}\cos 6x + \frac{6}{32}\cos 4x - \frac{15}{32}\cos 2x + \frac{10}{32} \right) dx$$

$$\begin{aligned}
&= \frac{1}{6} \left[ -\frac{1}{32 \times 6} \sin 6x + \frac{6}{32 \times 4} \sin 4x - \frac{15}{32 \times 2} \sin 2x + \frac{10}{32} x \right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{6 \times 32} \left[ -\frac{1}{6} \sin 6x + \frac{3}{2} \sin 4x - \frac{15}{2} \sin 2x + 10x \right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{6 \times 32} \left( -\frac{1}{6} \sin \left( 6 \frac{\pi}{4} \right) + \frac{3}{2} \sin \left( 4 \frac{\pi}{4} \right) - \frac{15}{2} \sin \left( 2 \frac{\pi}{4} \right) + 10 \times \frac{\pi}{4} \right) \\
&= \frac{1}{6 \times 32} \left( -\frac{1}{6} \sin \frac{3\pi}{2} + \frac{3}{2} \sin \pi - \frac{15}{2} \sin \frac{\pi}{2} + \frac{5\pi}{2} \right) \\
&= \frac{1}{6 \times 32} \left( \frac{1}{6} - \frac{15}{2} + \frac{5\pi}{2} \right) \\
&= \frac{1}{6 \times 32} \left( \frac{1}{6} - \frac{45}{6} + \frac{15\pi}{6} \right) \\
&= \frac{1}{6 \times 32} \left( \frac{15\pi - 44}{6} \right) \\
&= \frac{1}{6 \times 32 \times 6} \times \frac{15\pi - 44}{1} \\
&= \frac{15\pi - 44}{1152}
\end{aligned}$$

Conclusion :

$$\int_0^{\frac{\pi}{4}} \left( \int_0^x \sin t \cos t dt \right) dx = \frac{15\pi - 44}{1152}$$

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PLEG – Informaticien

Lycée Bilingue de Nylon Brazzaville Douala -  
Cameroun