Exercices corrigés.

Montrer que $\int_0^{\frac{\pi}{4}}\left(\int_0^x \sin t\cos t dt\right)dx=\frac{15\pi-44}{1152}$

Résolution

Posons $I=\int_0^{\frac{\pi}{4}}\left(\int_0^x \sin t\cos t dt\right)dx=\frac{15\pi-44}{1152}$

Considérons l'intégrale $J=\int_0^x \sin t\cos t dt$

$J=\int_0^x \sin^5 t\cos t dt=\left[\frac{\sin^6 t}{6}\right]_0^x=\frac{\sin^6 x}{6}$, car $\sin^5 t\cos t dt$ est de la forme $u'u^5$ dont une primitive est $\frac{u^6}{6}+k$

Donc $I=\int_0^{\frac{\pi}{4}}\frac{\sin^6 x}{6}dx=\frac{1}{6}\int_0^{\frac{\pi}{4}}\sin^6 x dx$

Linéarisons $\sin^6 x$

Posons $z=\cos\theta+i\sin\theta$

$$\sin\theta=\frac{z-\bar{z}}{2i}\Rightarrow\sin^6\theta=\left(\frac{z-\bar{z}}{2i}\right)^6=\left(\frac{1}{2i}\right)^6(z-\bar{z})^6$$

Développons $(z-\bar{z})^6$

$$\begin{aligned}(z-\bar{z})^6=(z+(-\bar{z}))^6&=z^6+6z^5(-\bar{z})+15z^4(-\bar{z})^2+20z^3(-\bar{z})^3+15z^2(-\bar{z})^4+6z(-\bar{z})^5+(-\bar{z})^6\\&=z^6+\bar{z}^6+6z\bar{z}\left(z^4+\bar{z}^4\right)+15(z\bar{z})^2\left(z^2+\bar{z}^2\right)+20z^3(-\bar{z})^3\\&=2\cos 6\theta-12\cos 4\theta+30\cos 2\theta-20\end{aligned}$$

D'om on tire que $\sin^6 x=-\frac{1}{64}(2\cos 6x-12\cos 4x+30\cos 2x-20)$

$$=-\frac{1}{32}\cos 6x+\frac{12}{64}\cos 4x-\frac{30}{64}\cos 2x+\frac{20}{64}$$

$$=-\frac{1}{32}\cos 6x+\frac{6}{32}\cos 4x-\frac{15}{32}\cos 2x+\frac{10}{32}$$

$$I=\frac{1}{6}\int_0^{\frac{\pi}{4}}\sin^6 x dx=\frac{1}{6}\int_0^{\frac{\pi}{4}}\left(-\frac{1}{32}\cos 6x+\frac{6}{32}\cos 4x-\frac{15}{32}\cos 2x+\frac{10}{32}\right)dx$$

$$=\frac{1}{6}\left[-\frac{1}{32\times 6}\sin 6x+\frac{6}{32\times 4}\sin 4x-\frac{15}{32\times 2}\sin 2x+\frac{10}{32}x\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{6\times 32}\left[-\frac{1}{6}\sin 6x+\frac{3}{2}\sin 4x-\frac{15}{2}\sin 2x+10x\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{6\times 32}\left(-\frac{1}{6}\sin\left(6\frac{\pi}{4}\right)+\frac{3}{2}\sin\left(4\frac{\pi}{4}\right)-\frac{15}{2}\sin\left(2\frac{\pi}{4}\right)+10\times\frac{\pi}{4}\right)$$

$$=\frac{1}{6\times 32}\left(-\frac{1}{6}\sin\frac{3\pi}{2}+\frac{3}{2}\sin\pi-\frac{15}{2}\sin\frac{\pi}{2}+\frac{5\pi}{2}\right)$$

$$=\frac{1}{6\times 32}\left(\frac{1}{6}-\frac{15}{2}+\frac{5\pi}{2}\right)$$

$$=\frac{1}{6\times 32}\left(\frac{1}{6}-\frac{45}{6}+\frac{15\pi}{6}\right)$$

$$=\frac{1}{6\times 32}\left(\frac{15\pi-44}{6}\right)$$

$$=\frac{1}{6\times 32\times 6}\times\frac{15\pi-44}{1}$$

$$=\frac{15\pi-44}{1152}$$

Conclusion :

$$\int_0^{\frac{\pi}{4}}\left(\int_0^x \sin t\cos t\,dt\right)dx=\frac{15\pi-44}{1152}$$

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