Exercices corrigés : Intégration.

Le but de cet exercice est de calculer $I=\int_0^{\frac{\pi}{4}}\frac{dx}{\cos^5 x}$

On pose pour tout entier naturel n, $I_n=\int_0^{\frac{\pi}{4}}\frac{dx}{\cos^{2n+1} x}$

1. Trouver deux réels a et b tels que pour tout réel x de $\left[0;\frac{\pi}{4}\right]$:

   $$\frac{1}{\cos x}=\frac{a\cos x}{1-\sin x}+\frac{b\sin x}{1+\sin x}$$

   Déduisez –en la valeur de I_0

2. En intégrant par parties, montrez que pour tout $n\geq 1$, $2nI_n=(2n-1)I_{n-1}+\frac{2^n}{\sqrt{2}}$

   Déduisez la valeur de I

Correction

1.

$$\frac{1}{\cos x}=\frac{\cos x}{\cos^2 x}=\frac{\cos x}{1-\sin^2 x}=\frac{\cos x}{(1-\sin x)(1+\sin x)}=\cos x\times\frac{1}{(1-\sin x)(1+\sin x)}$$

Considérons l'expression $\frac{1}{(1-\sin x)(1+\sin x)}$

Posons $X=\sin x$

$$\frac{1}{(1-\sin x)(1+\sin x)}=\frac{1}{(1-X)(1+X)}$$

Décomposons $\frac{1}{(1-X)(1+X)}$

$\frac{1}{(1-X)(1+X)}=\frac{a}{1-X}+\frac{b}{1+X}$, où a et b sont des réels à déterminer

Déterminons a et b

$$\frac{1}{(1+X)}=\frac{(1-X)a}{1-X}+\frac{(1-X)b}{1+X}=a+\frac{(1-X)b}{1+X}$$

En prenant $x=1$, on a $a=\frac{1}{(1+1)}=\frac{1}{2}$

$$\frac{1}{(1-X)}=\frac{(1+X)a}{1-X}+\frac{(1+X)b}{1+X}=\frac{(1+X)a}{1-X}+b$$

En posant $x=-1$, on a $b=\frac{1}{(1+1)}=\frac{1}{2}$

$a = b = \frac{1}{2}$

On obtient $\frac{1}{(1-X)(1+X)} = \frac{1}{2}\left(\frac{1}{1-X} + \frac{1}{1+X}\right)$

On peut donc écrire $\frac{1}{(1-\sin x)(1+\sin x)} = \frac{1}{2}\left(\frac{1}{1-\sin x} + \frac{1}{1+\sin x}\right)$

Et $\frac{\cos x}{(1-\sin x)(1+\sin x)} = \frac{1}{2}\left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}\right)$

Conclusion $\frac{1}{\cos x} = \frac{1}{2}\left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}\right)$

Déduisons la valeur de I_0

$$I_0 = \int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}\right)dx$$

$$= \frac{1}{2}\left[-\ln(1-\sin x) + \ln(1+\sin x)\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[-\ln\left(1-\frac{\sqrt{2}}{2}\right) + \ln\left(1+\frac{\sqrt{2}}{2}\right)\right]..$$

$$= \frac{1}{2}\left[-\ln\left(\frac{2-\sqrt{2}}{2}\right) + \ln\left(\frac{2+\sqrt{2}}{2}\right)\right] = \frac{1}{2}\left[-\ln\left(2-\sqrt{2}\right) + \ln 2 + \ln\left(2+\sqrt{2}\right) - \ln 2\right]$$

$$= \frac{1}{2}\left[-\ln\left(2-\sqrt{2}\right) + \ln\left(2+\sqrt{2}\right)\right] = \frac{1}{2}\left[\ln\left(2+\sqrt{2}\right) - \ln\left(2-\sqrt{2}\right)\right]$$

$$I_0 = \frac{1}{2}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}$$

2. En intégrant par parties, montrons que $2nI_n = (2n-1)I_{n-1} + \frac{2^n}{\sqrt{2}}$

$$I_n = \int_0^{\frac{\pi}{4}} \frac{dx}{\cos^{2n+1} x} = I_n = \int_0^{\frac{\pi}{4}} \frac{dx}{\cos^{2n-1} x \times \cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} \times \frac{1}{\cos^{2n-1} x}dx$$

Posons $\begin{cases} u'(x) = \frac{1}{\cos^2 x} \\ v(x) = \frac{1}{\cos^{2n-1} x} \end{cases} \Rightarrow \begin{cases} u(x) = \tan x \\ v'(x) = \frac{(2n-1)\sin x \cos^{2n-2} x}{\cos^{4n-2} x} \end{cases}$

$$I_n=\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}(2n-1)\times\frac{\sin x\cos^{2n-2}x}{\cos^{4n-2}x}\times\frac{\sin x}{\cos x}dx$$

$$=\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}(2n-1)\times\frac{\sin^2 x\cos^{2n-2}x}{\cos^{4n-1}x}dx$$

$$=\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}-(2n-1)\int_0^{\frac{\pi}{4}}\frac{\left(1-\cos^2 x\right)\cos^{2n-2}x}{\cos^{4n-1}x}dx$$

$$=\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}-(2n-1)\int_0^{\frac{\pi}{4}}\left(\frac{\cos^{2n-2}x}{\cos^{4n-1}x}-\frac{\cos^{2n}x}{\cos^{4n-1}x}\right)dx$$

$$=\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}-(2n-1)\int_0^{\frac{\pi}{4}}\left(\frac{1}{\cos^{2n+1}x}-\frac{1}{\cos^{2n-1}x}\right)dx$$

$$=\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}-(2n-1)\left(\underbrace{\int_0^{\frac{\pi}{4}}\frac{1}{\cos^{2n+1}x}dx}_{I_n}+\underbrace{\int_0^{\frac{\pi}{4}}\frac{1}{\cos^{2n-1}x}dx}_{I_{n-1}}\right)$$

$$=\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}-(2n-1)\left(I_n-I_{n-1}\right)$$

On obtient finalement $I_n=\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}-(2n-1)\left(I_n-I_{n-1}\right)$

$$I_n=\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}-(2n-1)I_n+(2n-1)I_{n-1}$$

$$I_n+(2n-1)I_n=(2n-1)I_{n-1}+\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}$$

Soit

$$2nI_n=(2n-1)I_{n-1}+\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}$$

Calculons $\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}$

$$\left[\frac{1}{\cos^{2n-1}x}\tan x\right]_0^{\frac{\pi}{4}}=\frac{1}{\cos^{2n-1}\frac{\pi}{4}}\tan\frac{\pi}{4}-\frac{1}{\cos^{2n-1}0}\tan 0=\frac{1}{\cos^{2n-1}\frac{\pi}{4}}\tan\frac{\pi}{4}=\frac{1}{\cos^{2n-1}\frac{\pi}{4}}$$

$$\frac{1}{\cos^{2n-1}\frac{\pi}{4}}=\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^{2n-1}}=\left(\frac{2}{\sqrt{2}}\right)^{2n-1}=\left(\sqrt{2}\right)^{2n-1}=\frac{\left(\sqrt{2}\right)^{2n}}{\sqrt{2}}=\frac{\left(\left(\sqrt{2}\right)^{2}\right)^{n}}{\sqrt{2}}=\frac{2^n}{\sqrt{2}}$$

Conclusion : $2nI_n=(2n-1)I_{n-1}+\frac{2^n}{\sqrt{2}}(*)$

Déduisons : $I=\int_0^{\frac{\pi}{4}}\frac{dx}{\cos^{2n+1}x}=I_5$

La relation (*) est une relation de récurrence entre les termes.

Nous allons calculer les termes successifs jusqu'au 6 ieme terme

$$2I_1=I_0+\frac{\sqrt{2}}{2}=I_0+\sqrt{2}\Rightarrow I_1=\frac{1}{2}\left(I_0+\sqrt{2}\right)$$

$$I_1=\frac{1}{2}\left(\frac{1}{2}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\sqrt{2}\right)=\frac{1}{4}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{\sqrt{2}}{2}$$

$$4I_2=3I_1+\frac{2^2}{\sqrt{2}}=3I_1+2\sqrt{2}\Rightarrow I_2=\frac{1}{4}\left(3I_1+2\sqrt{2}\right)$$

$$I_2=\frac{1}{4}\left[\frac{3}{4}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{3\sqrt{2}}{2}+2\sqrt{2}\right]=\frac{1}{4}\left[\frac{3}{4}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{7\sqrt{2}}{2}\right]$$

$$I_2=\frac{3}{16}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{7\sqrt{2}}{8}$$

$$6I_3=5I_2+\frac{8}{\sqrt{2}}\Rightarrow I_3=\frac{1}{6}\left(5I_2+4\sqrt{2}\right)$$

$$I_3=\frac{1}{6}\left(5I_2+4\sqrt{2}\right)=\frac{1}{6}\left(\frac{15}{16}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{35\sqrt{2}}{8}+4\sqrt{2}\right)=\frac{1}{6}\left(\frac{15}{16}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{67\sqrt{2}}{8}\right)$$

$$I_3=\frac{15}{96}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{67\sqrt{2}}{48}$$

$$8I_4=7I_3+\frac{4^2}{\sqrt{2}}=7I_3+8\sqrt{2}\Rightarrow I_4=\frac{1}{8}\left(7I_3+8\sqrt{2}\right)$$

$$I_4=\frac{1}{8}\left(\frac{105}{96}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{469\sqrt{2}}{48}+8\sqrt{2}\right)=\frac{1}{8}\left(\frac{105}{96}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{853\sqrt{2}}{48}\right)$$

$$I_4=\frac{105}{768}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{853\sqrt{2}}{384}$$

$$10I_5=9I_4+\frac{2^5}{\sqrt{2}}\Rightarrow I_5=\frac{1}{10}\left(9I_4+16\sqrt{2}\right)$$

$$I_5=\frac{1}{10}\left(9I_4+16\sqrt{2}\right)=\frac{1}{10}\left(\frac{945}{768}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{7767\sqrt{2}}{384}+16\sqrt{2}\right)$$

$$I_5=\frac{945}{7680}\ln\frac{\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)}+\frac{13911\sqrt{2}}{3840}$$

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