Second International Olympiad, 1960

1960/1.

Determine all three-digit numbers N having the property that N is divisible by 11, and $N/11$ is equal to the sum of the squares of the digits of N.

1960/2.

For what values of the variable x does the following inequality hold:

$$\frac{4x^2}{(1-\sqrt{1+2x})^2} < 2x+9?$$

1960/3.

In a given right triangle ABC, the hypotenuse BC, of length a, is divided into n equal parts (n an odd integer). Let α be the acute angle subtending, from A, that segment which contains the midpoint of the hypotenuse. Let h be the length of the altitude to the hypotenuse of the triangle. Prove:

$$\tan\alpha = \frac{4nh}{(n^2-1)a}.$$

1960/4.

Construct triangle ABC, given h_a, h_b (the altitudes from A and B) and m_a, the median from vertex A.

1960/5.

Consider the cube $ABCDA'B'C'D'$ (with face $ABCD$ directly above face $A'B'C'D'$).
(a) Find the locus of the midpoints of segments XY, where X is any point of AC and Y is any point of $B'D'$.
(b) Find the locus of points Z which lie on the segments XY of part (a) with $ZY = 2XZ$.

1960/6.

Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. Let V_1 be the volume of the cone and V_2 the volume of the cylinder.
(a) Prove that $V_1 \neq V_2$.
(b) Find the smallest number k for which $V_1 = kV_2$, for this case, construct the angle subtended by a diameter of the base of the cone at the vertex of the cone.

1960/7.
An isosceles trapezoid with bases a and c and altitude h is given.
(a) On the axis of symmetry of this trapezoid, find all points P such that both legs of the trapezoid subtend right angles at P.
(b) Calculate the distance of P from either base.
(c) Determine under what conditions such points P actually exist. (Discuss various cases that might arise.)