Fourth International Olympiad, 1962

1962/1.

Find the smallest natural number n which has the following properties:
(a) Its decimal representation has 6 as the last digit.
(b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

1962/2.

Determine all real numbers x which satisfy the inequality:

$$\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}.$$

1962/3.

Consider the cube $ABCDA'B'C'D'$ ($ABCD$ and $A'B'C'D'$ are the upper and lower bases, respectively, and edges AA', BB', CC', DD' are parallel). The point X moves at constant speed along the perimeter of the square $ABCD$ in the direction $ABCDA$, and the point Y moves at the same rate along the perimeter of the square $B'C'CB$ in the direction $B'C'CBB'$. Points X and Y begin their motion at the same instant from the starting positions A and B', respectively. Determine and draw the locus of the midpoints of the segments XY.

1962/4.

Solve the equation $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

1962/5.

On the circle K there are given three distinct points A, B, C. Construct (using only straightedge and compasses) a fourth point D on K such that a circle can be inscribed in the quadrilateral thus obtained.

1962/6.

Consider an isosceles triangle. Let r be the radius of its circumscribed circle and ρ the radius of its inscribed circle. Prove that the distance d between the centers of these two circles is

$$d = \sqrt{r(r-2\rho)}.$$

1962/7.

The tetrahedron $SABC$ has the following property: there exist five spheres, each tangent to the edges $SA, SB, SC, BCCA, AB$, or to their extensions.
(a) Prove that the tetrahedron $SABC$ is regular.
(b) Prove conversely that for every regular tetrahedron five such spheres exist.